[11] Ramnath, R. V. and Sandri, G., "A Generalized Multiple Scales Approach to a Class of Linear Differential Equations," *Journal of Mathematical Analysis and Applications*, Vol. 28, No. 2, Nov. 1969, pp. 339–364.

[12] Ramnath, R. V., "A New Analytical Approximation to the Thomas-Fermi Model in Atomic Physics," *Journal of Mathematical Analysis and Applications*, to be published.

[13] Ramnath, R. V., "On A Class of Nonlinear Differential Equations of Astrophysics," *Journal of Mathematical Analysis and Applications*, to be published.

[14] Klimas, A., Ramnath, R. V., and Sandri, G., "On the Compatibility Problem for the Uniformization of Asymptotic Expansions," *Journal of Mathematical Analysis and Applications*, to be published.

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Optimal Two- and Three-Impulse Fixed-Time Rendezvous in the Vicinity of a Circular Orbit

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Minimum-fuel, multiple-impulse orbital rendezvous is investigated for the case in which the transfer time is specified (time-fixed case). A method for obtaining optimal solutions is employed which is applicable to rendezvous or orbit transfer between elliptical orbits of low eccentricity. In this method optimal solutions are constructed by satisfying the necessary conditions for the primer vector. It is assumed that the terminal orbits lie close enough to an intermediate circular reference orbit that the linearized equations of motion can be used to describe the transfer. The linear boundary value problem for the impulse magnitudes for rendezvous is then solved analytically. As an application of the method, optimal two- and three-impulse fixed-time rendezvous transfers between coplanar circular orbits are obtained for a range of transfer times. These linearized solutions combined with previously obtained four-impulse solutions provide a complete solution for fixed-time coplanar circle-to-circle rendezvous between close orbits for transfer times up to nearly two terminal orbit periods.

Nomenclature

$(\dot{\ })$ = first derivative of ( ) with respect to time. Time derivatives of vectors are taken with respect to an inertial reference frame
$(\ )'$ = first derivative with respect to dimensionless time τ
$|(\ )|$ = determinant of the array ( )
a = radius of reference circular orbit
$A\#,B\#,C\#,D\#$ = arbitrary constants in Sec. V
B_{Fj} = partition of state transition matrix (Eq. 38)
b = variable defined by Eq. (31)
c = $\cos\theta_2$
$\mathbf{h}_j$ = vector column of H matrix
H = matrix defined by Eq. (3)
I = identity matrix
k = variable defined by Eq. (11)
k_j = variable defined by Eq. (33)
m = variable defined by Eq. (10)
m_j = variable defined by Eq. (32)
N_{2j} = partition of transition matrix [Eq. (40)]
$\mathbf{P}$ = primer vector
q_j = variable defined by Eq. (12)
$\mathbf{r}$ = position vector
δR = nondimensional difference between final and initial circular orbit radii
s = $\sin\theta_2$
t = time
T_{Fj} = partition of transition matrix [Eq. (40)]
$\mathbf{u}_j$ = unit vector in direction of jth thrust impulse
$\mathbf{v}$ = velocity vector
$\Delta\mathbf{V}_j$ = vector velocity change due to jth thrust impulse
$\Delta\mathbf{V}$ = vector defined by Eq. (1)
$\mathbf{w}_j$ = vector column of W matrix
W = matrix defined by Eq. (1)
$\mathbf{x}$ = position-velocity state vector
β = initial phase angle of target (Fig. 12)
$\delta(\ )$ = first variation of ( )
$\Delta(\ )$ = change in ( )
θ = central angle
θ_2 = $\tau_2 - \tau_1$ (Sec. V)
λ = radial component of primer vector
μ = circumferential component of primer vector
$\rho(\ )$ = rank of ( )
τ = dimensionless time (see Appendix)
Φ_{ji} = state transition matrix between times t_i and t_j
ω = mean motion of reference orbit

Subscripts

0 = initial value
F = final value
H = half-transfer time [Eq. (3)]
j = time of jth impulse
r = radial component
θ = circumferential component

Superscript

$+$ = initial coast if precedes variable; final coast if succeeds variable

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I. Introduction

IN the study of minimum fuel trajectories in an inverse square gravitational field, considerable attention has been directed to an aspect of the problem which has been called

Lawden's Problem.[1] Lawden's Problem assumes a variable thrust rocket capable of delivering impulses (having constant exhaust velocity and unbounded thrust magnitude).

In papers by Edelbaum,[1] Gobetz and Doll,[2] and Robinson,[3] the known solutions to Lawden's Problem are discussed. Many important solutions have been obtained, predominantly for the time-open case, in which the transfer time is unspecified. Time-fixed solutions which have been reported include the work of Lion and Handelsman,[4] Marec,[5,6] and Prussing.[7]

The objective of this investigation is to extend the results of Ref. 7 by applying the same general method to obtain characteristics of minimum fuel two- and three-impulse thrust programs for rendezvous in the vicinity of a circular orbit. For the specific case of rendezvous between close, coplanar, circular orbits, analytical solutions for the optimal thrust programs are obtained for a range of fixed transfer times. These solutions, combined with the four-impulse results of Ref. 7, constitute the essential results of Ref. 10 and provide a complete solution for optimal fixed-time multiple-impulse rendezvous between close, coplanar, circular orbits for transfer times up to nearly two terminal orbit periods. These linearized results offer some physical insight into the nature of multiple-impulse solutions and provide approximate initial conditions which could be used in an iterative solution to the corresponding nonlinear problem.

II. Description of the Method

The method employed to obtain optimal multiple-impulse thrust programs is based on satisfying the set of necessary conditions first derived by Lawden.[8] These conditions are conveniently expressed in terms of the primer vector, the vector of adjoint variables associated with the vehicle velocity vector. Details of the method used to construct primer vector solutions which satisfy the required necessary conditions are given in Ref. 7.

In that reference optimal multiple-impulse rendezvous between close, coplanar, low-eccentricity orbits is discussed. A linearized analysis is made using an intermediate circular orbit as a reference orbit. A technique is presented by which primer vector solutions which satisfy Lawden's necessary conditions can be constructed for a specified number of impulses. Once a primer vector solution has been so constructed, the optimal directions and times of application of the impulses are determined for the fixed-time rendezvous. The additional information necessary to completely characterize the thrust program, the size of each thrust impulse, is determined by the analytical solution of a linear boundary value problem. In Ref. 7 the solution is carried out for the case of four-impulse, circle-to-circle rendezvous.

III. Rendezvous Boundary Value Problem

Using the notation of Ref. 7, the boundary value equation to determine the sizes of the thrust impulses can be written as,

$$\widetilde{\delta \mathbf{x}}_F \triangleq \delta \mathbf{x}_F - \Phi_{F0}\delta \mathbf{x}_0 = W\mathbf{\Delta V} \quad (1)$$

where $\delta \mathbf{x}$ is the six-component position-velocity state variation (the contemporaneous vector difference between the actual state and the circular reference orbit state). The subscripts F and 0 refer to the final and initial times, respectively, (for rendezvous the final state variation of the transfer vehicle is the state variation of the target body). Φ_{F0} is the state transition matrix for the linearized system between the specified initial time, τ_0, and the final time, τ_F.

In Eq. (1) $\mathbf{\Delta V}$ is the vector of unknowns, the magnitudes of the instantaneous velocity changes caused by the thrust impulses. The W matrix contains the information gained from the primer vector construction, namely, the optimal times and directions of the thrust impulses. The jth column of the matrix is the vector representing the change in $\delta \mathbf{x}_F$ due to a unit magnitude velocity change made at the optimal time and in the optimal direction of the jth impulse. For a specified transfer time, the initial state variation and desired final state variation are known. The solution to the boundary value problem is the $\mathbf{\Delta V}$ which will accomplish the optimal rendezvous. As discussed in Ref. 7, only those $\mathbf{\Delta V}$ having non-negative components are admissible solutions.

In describing a coplanar rendezvous transfer, the state variation vector has four components—two of position variation and two of velocity variation. For a three-impulse coplanar rendezvous, the W matrix of Eq. (1) is 4×3. Therefore the equation possesses a unique solution for the unknown $\mathbf{\Delta V}$ if and only if the rank of the augmented W matrix and the rank of W are each equal to three. This requires that the augmented W matrix (4×4) be singular. Denoting the columns of W as vectors $\mathbf{w}_1$, $\mathbf{w}_2$, and $\mathbf{w}_3$, this condition can be expressed as

$$|\widetilde{\delta \mathbf{x}}_F \mathbf{w}_1 \mathbf{w}_2 \mathbf{w}_3| = 0 \quad (2)$$

Rendezvous with Terminal Coasting Periods

For a specified transfer time and specified initial states of the target and rendezvous vehicles, a coasting period in either the initial or final orbit (or both) may be fuel optimal. Since the fuel cost for an orbital maneuver generally depends strongly on the initial geometry of the target and rendezvous vehicles, a portion of the transfer time in some situations is best invested in a coasting period to allow a geometrically more favorable rendezvous. An initial coast implies waiting in the initial orbit; the first impulse is applied at a time later than the specified initial time. A final coast implies that the rendezvous actually occurs at a time earlier than the final time, resulting in a coasting period in the final orbit until the specified final time.

If an impulsive transfer is to be fuel-optimal, the magnitude of the primer vector must not exceed unity at any time during the transfer, including the coasting period.[7,8] Figure 1 shows such a magnitude, p, of the primer vector as a function of the dimensionless time τ (τ changes by 2π in one reference orbit period). This figure is a primer magnitude history for a circular reference orbit and was obtained using the method of Ref. 7. The necessary conditions[7,8] require that the impulses be applied at those times for which the primer magnitude is unity, namely, τ_1, τ_2, and τ_3 in Fig. 1. Since $p > 1$ for $\tau < \tau_0$, no initial coast is fuel-optimal for this primer solution; τ_0, the initial time, is also τ_1, the time of the first impulse. The specified transfer time for this example is $\tau_F - \tau_0$, requiring a final coast of duration $\tau_F - \tau_3$.

The dashed continuation of the primer magnitude for $\tau > \tau_F$ in Fig. 1 shows that this three-impulse solution with final coast is actually a portion of the type of primer vector solution examined in detail in Ref. 7, namely, a four-impulse solution for the circular reference orbit. This observation allows one to interpret an opt mal three-impulse rendezvous with final coast as an optimal four-impulse rendezvous for which the fourth impulse has zero magnitude. Three-impulse solutions with an initial coast period can be interpreted analogously.

Based on these considerations, the primer vector solutions for optimal three-impulse rendezvous with a terminal coast can be inferred directly from the four-impulse solutions of Ref. 7. For a given optimal four-impulse primer solution, the final coast solution shown in Fig. 1 is valid for all fixed transfer times $\tau_F - \tau_0$ for which $\tau_F \epsilon [\tau_3, \tau_4]$. The analogous initial coast solution is valid for all $\tau_F - \tau_0$ for which $\tau_0 \epsilon [\tau_1, \tau_2]$. The boundary conditions and the impulse magnitudes for these three-impulse rendezvous differ from those of the corresponding four-impulse rendezvous and must be determined using Eq. (1).

IV. Optimal Three-Impulse Circle-to-Circle Coplanar Rendezvous

Solutions with Terminal Coasts

To directly supplement the results of Ref. 7, the case of circle-to-circle coplanar rendezvous is examined. As in Sec. VI of Ref. 7, it is convenient to define

$$H \triangleq \Phi_{HF} W \tag{3}$$

where H, in this case, is 4×3 and τ_H is the half-transfer time, $(\tau_0 + \tau_F)/2$, of a given four-impulse solution. Equation (1) can then be expressed as

$$\widetilde{\delta \mathbf{x}}_H = \delta \mathbf{x}_H - \Phi_{H0}\delta \mathbf{x}_0 = H\mathbf{\Delta V} \tag{4}$$

where

$$\delta \mathbf{x}_H = \Phi_{HF}\delta \mathbf{x}_F \tag{5}$$

Note that if any thrust impulses occur after the half-transfer time, $\delta \mathbf{x}_H$ is not the state variation of the vehicle at time τ_H, but rather the final state variation (including the effect of all thrusts) transformed back to the time τ_H. On the other hand, since the final state variations of the vehicle and target body are equal for rendezvous, $\delta \mathbf{x}_H$ is the state variation of the (unthrusted) target body at the half-transfer time.

To distinguish between initial and final coasting periods, define

$$^+H \triangleq \Phi_{HF}{}^+W; \quad ^+W = [\mathbf{w}_2\mathbf{w}_3\mathbf{w}_4] \tag{6}$$

and

$$H^+ \triangleq \Phi_{HF}W^+; \quad W^+ = [\mathbf{w}_1\mathbf{w}_2\mathbf{w}_3] \tag{7}$$

where the $\mathbf{w}_j$ are the columns of the W matrix for a four-impulse primer solution. The $^+$ superscript prior to a variable denotes an initial coast; the superscript following a variable denotes final coast.

Since Φ_{HF} is nonsingular, it can be shown by Sylvester's inequality[9] that H and W have the same rank, which is three if the $\mathbf{w}_j$ are linearly independent. As mentioned in Ref. 7 the $\mathbf{w}_j$ were found to be linearly independent for four-impulse primer solutions which result in admissible circle-to-circle rendezvous. Therefore the condition for the existence of a unique three-impulse solution to Eq. (4) is obtained from Eq. (2) as,

for initial coast:

$$|\widetilde{\delta \mathbf{x}}_H \mathbf{h}_2\mathbf{h}_3\mathbf{h}_4| = 0 \tag{8}$$

for final coast:

$$|\widetilde{\delta \mathbf{x}}_H \mathbf{h}_1\mathbf{h}_2\mathbf{h}_3| = 0 \tag{9}$$

where the $\mathbf{h}_j$ are the columns of the H matrix for the four-impulse solutions of Ref. 7.

The components of $\widetilde{\delta \mathbf{x}}_H$ are given by Eq. (58) in the Appendix. Several auxiliary variables simplify the notation in evaluating the determinants of Eqs. 8 and 9. Let h_{ij} denote the ith component of the vector $\mathbf{h}_j$, and define

$$m \triangleq h_{11}h_{42} - h_{12}h_{41} \tag{10}$$

$$k \triangleq h_{22}h_{31} - h_{21}h_{32} \tag{11}$$

$$q_j \triangleq 2h_{4j} + 3h_{1j} \; (j = 1,2) \tag{12}$$

In terms of these variables and the components of $\widetilde{\delta \mathbf{x}}_H$ one can evaluate Eqs. (8) and (9). Since $\tau_F = \tau_4$ for the four-impulse case,

for initial coast

$$\delta\theta_4 = (q_2k/2h_{32}m)\delta R \tag{13}$$

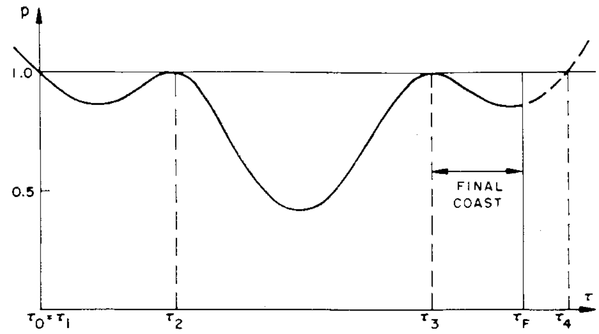

Fig. 1 Primer magnitude for optimal three-impulse-with-coast transfer.

for final coast

$$\delta\theta_4 = -(q_2k/2h_{32}m)\delta R \tag{14}$$

As one would expect, Eqs. (13) and (14) are identical to the conditions obtained from the four-impulse results of Ref. 7 if one sets $\Delta V_1 = 0$ and $\Delta V_4 = 0$, respectively. Unlike the four-impulse solutions, the values of $\delta\theta_4$ and δR are constrained to be proportional for a given primer vector solution in the three-impulse case.

The velocity changes necessary to perform rendezvous are obtained from Eq. (4) using Eqs. (13) and (14)

for initial coast

$$^+\Delta V_2 = (\delta R/4m)(q_2h_{31}/h_{32} - q_1) \tag{15}$$

$$^+\Delta V_3 = (\delta R/4m)(q_2h_{31}/h_{32} + q_1) \tag{16}$$

$$^+\Delta V_4 = -(q_2/2m)\delta R \tag{17}$$

The total cost (sum of the velocity changes) is then

$$^+\Delta V = (q_2/2m)(h_{31}/h_{32} - 1)\delta R \tag{18}$$

For final coast

$$\Delta V_1{}^+ = (q_2/2m)\delta R \tag{19}$$

$$\Delta V_2{}^+ = (\delta R/4m)(-q_2h_{31}/h_{32} - q_1) \tag{20}$$

$$\Delta V_3{}^+ = (\delta R/4m)(-q_2h_{31}/h_{32} + q_1) \tag{21}$$

$$\Delta V^+ = (q_2/2m)(1 - h_{31}/h_{32})\delta R \tag{22}$$

The velocity changes for the initial and final coast solutions are related, viz;

$$\Delta V_k{}^+ = -{}^+\Delta V_{5-k} \; (k = 1,2,3) \tag{23}$$

$$\Delta V^+ = -{}^+\Delta V \tag{24}$$

Thus it appears that, for example, if a final coast solution is admissible ($\Delta V_k{}^+ \geq 0$), the initial coast solution is not. However, there is a closely related initial coast transfer which is admissible. From Eq. (1) it is seen that by reversing the direction of all thrust impulses (by replacing $\mathbf{\Delta V}$ by $-\mathbf{\Delta V}$) while keeping the thrust times and magnitudes unchanged, one transfers a vehicle from an initial state $-\delta \mathbf{x}_0$ to a final state $-\delta \mathbf{x}_F$. Denoting this transfer as the reciprocal transfer, one observes that in the case of an admissible final coast solution, the reciprocal initial coast solution is also admissible, and vice-versa. To satisfy the boundary conditions one must change the algebraic signs of $\delta\theta_F$ and δR in Eqs. (8) and (9), but $|\delta\theta_F|$, $|\delta R|$, and the fuel cost are the same for both admissible optimal transfers.

In addition, since the primer vector satisfies a linear, homogeneous, second order differential equation involving no first derivative terms,[7,8] a solution of the primer vector equation run backwards in time is also a solution to the forward-time equation. This dual solution[10] represents a rendezvous in which the sign of δR is changed while $\delta\theta_F$ remains unchanged, for the same thrust magnitude. But here again $|\delta\theta_F|$, $|\delta R|$

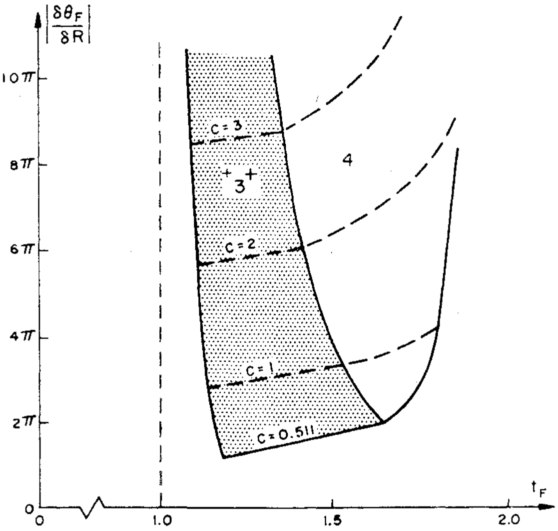

Fig. 2 Final state variations reached optimally by four and three-plus-coast solutions. C = $\Delta V/\delta R$.

and the fuel cost are the same. The dual solution, of course, also has a reciprocal solution.

The conditions for an admissible three-impulse solution with terminal coast differ slightly from those in the four-impulse case. As in the four-impulse case, $h_{31}/h_{32} < 0$ is one condition. In addition, another condition for admissibility is evident from Eqs. (15–17) and (19–21):

$$|q_2/h_{32}| > |q_1/h_{31}| \tag{25}$$

For the coplanar circle-to-circle case, the above conditions for an admissible three-impulse solution with terminal coast are satisfied along the lower-transfer-time boundary of the region of final state variations reached optimally by four-impulse rendezvous transfers. Figure 2 displays the four-impulse region results of Ref. 7 (the region labelled 4) along with the three-impulse with coast region (labelled $^+3^+$, denoting either a final or initial coasting period), which will be explained in more detail below. Due to the symmetry introduced by choosing the reference orbit to lie half-way between the terminal orbits, only $|\delta\theta_F/\delta R|$ need be plotted as a function of the transfer time t_F, measured in reference orbit periods (see Appendix for definition of symbols).

In Fig. 2 the final state variations for rendezvous are shown which are reachable by admissible three-impulse transfers with final coasting periods. The lines of constant dimensionless cost per radius change ($\Delta V/\delta R$) are straight in the $^+3^+$ region. Their slope reflects the rate at which $\delta\theta_F$ changes due to a terminal coast. For a final coast, $\tau_F \epsilon [\tau_3,\tau_4]$, and

$$\delta\theta_F/\delta R = \delta\theta_3/\delta R - \tfrac{3}{4}(\tau_F - \tau_3) \tag{26}$$

where $\delta\theta_3$ is the value of $\delta\theta$ at the time of the third impulse of the four-impulse primer solution. $\delta\theta_F$ and $\delta\theta_3$ are related by Eq. (26) since the final coast duration is $\tau_F - \tau_3$ and $\dot{\delta\theta} = -\frac{3}{4}\delta R$ in the final orbit (see Appendix).

Likewise, for an initial coast $\tau_0 \epsilon [\tau_1,\tau_2]$ and

$$\delta\theta_F/\delta R = \delta\theta_3/\delta R + \tfrac{3}{4}(\tau_2 - \tau_0) \tag{27}$$

since $\dot{\delta\theta} = \frac{3}{4}\delta R$ in the initial orbit.

The value $\delta\theta_3/\delta R$ forms the lower-transfer-time boundary of the $^+3^+$ region (zero duration terminal coast). The cost per radius change is constant along lines given by Eqs. (26) and (27) since the fuel cost of the transfer is not changed by a terminal coasting period. Since the magnitude, $|\delta\theta_F/\delta R|$, is plotted in Fig. 2, one must remember that during a final coast $\delta\theta_F/\delta R$ decreases [Eq. (26)], whereas during an initial coast it increases [Eq. (27)].

The point on the boundary of the four regions for which $|\delta\theta_F/\delta R|$ is a minimum was found to be characterized by

$$q_2/h_{32} = q_1/h_{31} \tag{28}$$

Referring to Eqs. (15–17) and (19–21), this implies that, for an initial coast $\Delta V_1 = \Delta V_2 = 0$, and, for a final coast $\Delta V_3 = \Delta V_4 = 0$. This implies that this point and the lower boundary of the $^+3^+$ region (along which the cost per radius change is 0.511) also form the upper boundary of a two-impulse transfer with the terminal coast.

Solutions with No Terminal Coasts

Figure 3 displays a primer magnitude time history which satisfies Lawden's necessary conditions for an optimal three-burn transfer, but does not allow a terminal coast [since $p > 1$ outside (τ_1,τ_3)]. This primer solution was constructed by the method of Ref. 7 and has no direct relationship with a four-impulse solution as did the terminal coast three-impulse solution.

The boundary value problem for this type of three-impulse solution is of the form of Eq. (1), where $\widetilde{\delta \mathbf{x}}_F$ is given in Eq. (57). Since there are no terminal coasts, $\tau_0 = \tau_1$ and $\tau_F = \tau_3$. The vector $\mathbf{w}_3$ (the third column of the W matrix) is the change in $\delta\mathbf{x}_F$ due to a unit impulse at the final time and is given by,

$$\mathbf{w}_3 = \begin{bmatrix} 0 \\ \mathbf{u}_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ u_{3r} \\ u_{3\theta} \end{bmatrix} \tag{29}$$

where $\mathbf{u}_3$ is the unit vector in the direction of the third impulse (and is therefore the primer vector).

Evaluating the determinant of Eq. (2), which is a necessary condition for the existence of a solution for the $\mathbf{\Delta V}$, one finds that

$$\widetilde{\delta\theta}_F \triangleq \delta\theta_F - \tfrac{3}{4}(\tau_F - \tau_0)\delta R = b\delta R \tag{30}$$

where

$$b \triangleq [u_{3r}(2k_4 + 3m_2) - 2k_3u_{3\theta}]/2(m_3u_{3\theta} - m_4u_{3r}) \tag{31}$$

and, by analogy with Eqs. (10) and (11)

$$m_j \triangleq w_{11}w_{j2} - w_{12}w_{j1} \tag{32}$$

$$k_j \triangleq w_{22}w_{j1} - w_{21}w_{j2} \tag{33}$$

To obtain a unique solution for $\mathbf{\Delta V}$, it is sufficient that one (3×3) submatrix of W be nonsingular and Eq. (30) be satisfied.

The submatrix formed by the first three rows of W, which has determinant value $u_{3r}m_2$, was found to be nonsingular in all cases examined. Using this result, the solution for $\mathbf{\Delta V}$

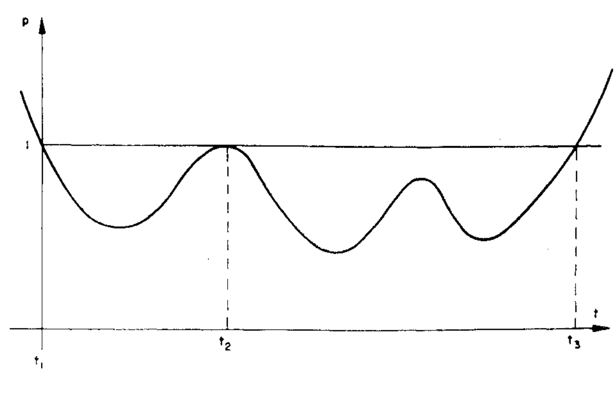

Fig. 3 Optimal three-impulse primer magnitude time history.

can be expressed as

$$\Delta V_1 = -[(bw_{12} - w_{22})/m_2]\delta R \tag{34}$$

$$\Delta V_2 = [(bw_{11} - w_{21})/m_2]\delta R \tag{35}$$

$$\Delta V_3 = -[(bm_3 + k_3)/u_{3r}m_2]\delta R \tag{36}$$

As in the terminal coast case, $\delta\theta_F$ and δR are proportional [Eq. (30)] for a given primer solution and the velocity changes can be expressed in terms of one independent state variable, δR.

The admissibility condition ($\Delta V_i \geq 0$, $i = 1,2,3$) is quite restrictive in comparison with the four-impulse case, in which $\delta\theta_F$ and δR could be specified independently for a given transfer time. Four-impulse solutions existed along vertical lines in the plane of reachable final state variations (Fig. 2). The admissibility condition restricted the solutions to be in certain segments of these vertical lines.

However, in the three-impulse case $\delta\theta_F$ is proportional to δR for a given transfer time. This corresponds to a point in the plane of reachable states rather than a line. The admissibility conditions determine whether or not this point is admissible; no range of values is allowed. This restriction, however, tends to be outweighed by the fact that there are many more three-impulse primer solutions which satisfy the necessary conditions than there are four-impulse solutions for a given range of transfer times. This results in the large part of the plane of reachable states occupied by admissible three-impulse solutions shown in Fig. 4.

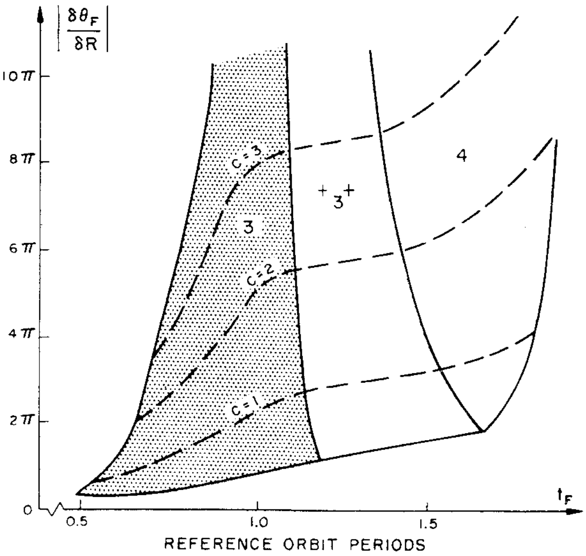

Fig. 4 Final state variations reached optimally by three- and four-impulse solutions.

V. Optimal Two-Impulse Rendezvous

For nonintersecting terminal orbits at least two impulses are required to perform rendezvous. In fact, for the coplanar circle-to-circle case considered, intersection of the terminal orbits implies intersection everywhere, and at least two impulses are always required to rendezvous (except in the trivial case $\delta\theta_F = 0$, in which no impulses are required).

In the coplanar case the four velocity change components (two for each impulse) are determined to satisfy the four desired final values of the state variations (two of position and two of velocity). For the linearized two-impulse case this offers a simplification, since, for a given transfer time and boundary conditions and no terminal coasts the problem has no extra degrees of freedom to be optimized. Whether or not the unique solution is optimal is determined by evaluating the primer vector corresponding to the solution and noting if it satisfies the necessary conditions. In the two-impulse case considered here, this method was found to be more expedient than constructing primer solutions using the method of Ref. 7.

The two-impulse boundary value problem can be written in the form

$$\widetilde{\delta \mathbf{x}}_F = \left[B_{F1} \vdots B_{F2} \right] \begin{bmatrix} \Delta\mathbf{V}_1 \\ \Delta\mathbf{V}_2 \end{bmatrix} \tag{37}$$

Here,

$$B_{Fj} = \Phi_{Fj}\left[\frac{0}{I}\right] \tag{38}$$

For a $2k \times 2k\Phi_{Fj}$, 0 and I are $k \times k$ zero and identity matrices, respectively.

Since $\widetilde{\delta \mathbf{x}} = \Phi_{2F}\widetilde{\delta \mathbf{x}}_F$,

$$\widetilde{\delta x}_2 = \left[B_{21} \vdots B_{22} \right] \begin{bmatrix} \Delta\mathbf{V}_1 \\ \Delta\mathbf{V}_2 \end{bmatrix} \tag{39}$$

Partitioning B_{2j} into $k \times k$ partitions,

$$B_{2j} = \left[\frac{N_{2j}}{T_{2j}}\right] \qquad (j = 1,2) \tag{40}$$

$$\widetilde{\delta \mathbf{x}}_2 = \begin{bmatrix} N_{21} & 0 \\ T_{21} & I \end{bmatrix} \begin{bmatrix} \Delta\mathbf{V}_1 \\ \Delta\mathbf{V}_2 \end{bmatrix} \tag{41}$$

where Eq. (38) has been used to evaluate B_{22}. Eq. (41) represents the boundary value problem to be solved for $\Delta\mathbf{V}_1$ and $\Delta\mathbf{V}_2$. Since the coefficient matrix is square, a unique solution for the velocity changes exists for every $\delta\mathbf{x}_2$ if and only if the coefficient matrix is nonsingular. One can easily show (e.g., by a Laplace expansion on the last column) that the determinant of the coefficient matrix is $-|N_{21}|$.

If N_{21} is nonsingular, the unique solution for the velocity changes is:

$$\begin{bmatrix} \Delta\mathbf{V}_1 \\ \Delta\mathbf{V}_2 \end{bmatrix} = \begin{bmatrix} N_{21}{}^{-1} & 0 \\ -T_{21}N_{21}{}^{-1} & I \end{bmatrix} \left[\widetilde{\delta \mathbf{x}}_2 \right] \tag{42}$$

Circle-to-Circle Coplanar Solutions

For circular terminal orbits the formula for $\widetilde{\delta \mathbf{x}}_2$ is similar to Eq. (57) of the Appendix, viz.,

$$\widetilde{\delta \mathbf{x}}_2{}^T = [\delta R \vdots \delta\theta_2 - \tfrac{3}{4}(\tau_2 - \tau_1)\delta R \vdots 0 \vdots -\tfrac{3}{2}\delta R] \tag{43}$$

For a circular reference orbit the elements of the state transition matrix are well-documented.[11] In terms of $\theta_2 \triangleq \tau_2 - \tau_1$,

$$|N_{21}| = 8(1 - \cos\theta_2) - 3\theta_2 \sin\theta_2 \tag{44}$$

This determinant vanishes for values of θ_2 which are multiples of 2π. The only other value of θ_2 in the interval $(0,4\pi)$ for which the determinant vanishes is 2.82π.[10]

For nonsingular N_{21} the velocity changes are

$$\begin{bmatrix} \Delta\mathbf{V}_1 \\ \\ \Delta\mathbf{V}_2 \end{bmatrix} = \frac{1}{|N_{21}|} \begin{bmatrix} 4s - 3\theta_2 & -2(1-c) & 0 & 0 \\ 2(1-c) & s & 0 & 0 \\ 3\theta_2 c - 4s & -2(1-c) & 8(1-c) - 3\theta_2 s & 0 \\ 14(1-c) - 6\theta_2 s & -s & 0 & 8(1-c) - 3\theta_2 s \end{bmatrix} \widetilde{\delta \mathbf{x}}_2 \tag{45}$$

where $s \triangleq \sin\theta_2$; $c \triangleq \cos\theta_2$.

Primer Vectors for Two-Impulse Solutions

From the velocity changes of Eq. (45) the corresponding primer vector solutions can be checked against the necessary conditions. Denoting the radial and tangential components of the primer vector by λ and μ, respectively, the time

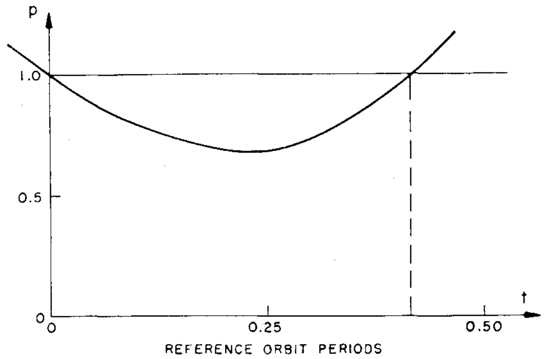

Fig. 5 Primer magnitude for optimal two-impulse rendezvous.

history of these components is given by[7]

$$\lambda(\theta) = A\# \cos\theta + B\# \sin\theta + 2C\# \tag{46}$$

$$\mu(\theta) = 2B\# \cos\theta - 2A\# \sin\theta - 3C\#\theta + D\# \tag{47}$$

where $\theta \triangleq \tau - \tau_1$ and $A\#$, $B\#$, $C\#$, and $D\#$ are arbitrary constants.

Since, along an optimal solution, the primer vector at the thrust time is the unit vector in the thrust direction, one attempts to satisfy the necessary conditions by setting

$$\left.\begin{aligned} \lambda_j &= \Delta V_{j_r}/\Delta V_j \quad (48)\\ \mu_j &= \Delta V_{j\theta}/\Delta V_j \quad (49) \end{aligned}\right\} \quad j = 1,2$$

This provides a sufficient number of boundary conditions on Eqs. (46) and (47) to evaluate the arbitrary constants. Since the first impulse occurs at $\tau = \tau_1(\theta = 0)$ and the second at $\tau = \tau_2(\theta = \theta_2)$, direct substitution into Eqs. (46) and (47) yields

$$\begin{bmatrix} \lambda_1 \\ \mu_1 \\ \lambda_2 \\ \mu_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 2 & 0 & 1 \\ c & s & 2 & 0 \\ -2s & 2c & -3\theta_2 & 1 \end{bmatrix} \begin{bmatrix} A\# \\ B\# \\ C\# \\ D\# \end{bmatrix} \tag{50}$$

Inversion of the above matrix yields the arbitrary constants in terms of the known primer vector components. The determinant of this matrix is $|N_{21}|$, and is given by Eq. (45)

$$\begin{bmatrix} A\# \\ B\# \\ C\# \\ D\# \end{bmatrix} = \frac{1}{|N_{21}|} \begin{bmatrix} 4(1-c) - 3\theta_2 s & 2s & -4(1-c) & -2s \\ 3\theta_2 c - 4s & 2(1-c) & 4s - 3\theta_2 & -2(1-c) \\ 2(1-c) & -s & 2(1-c) & s \\ 8s - 6\theta_2 c & 4(1-c) - 3\theta_2 s & 6\theta_2 - 8s & 4(1-c) \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \mu_1 \\ \lambda_2 \\ \mu_2 \end{bmatrix} \tag{51}$$

Evaluation of Eqs. (46) and (47) as a function of θ in the interval $[0,\theta_2]$ determines whether or not the two-impulse solution is optimal. If $\lambda^2 + \mu^2$ exceeds unity anywhere in the interval, the solution is nonoptimal.[8] Figure 5 displays an example primer magnitude for an optimal two-impulse solution with no terminal coast. Figure 6 shows the required form for a final coast solution. The final state variations reached optimally by 2 and $^+2^+$ transfers are shown in Fig. 7. Also displayed in Fig. 7 is the wedge-shaped region of final state variations reached by a linearized Hohmann transfer discussed in Ref. 7. Although three- and four-impulse transfers exist in this region, they can always be accomplished for the same fuel cost by a (two-impulse) Hohmann transfer having initial and final coasts.[7,10]

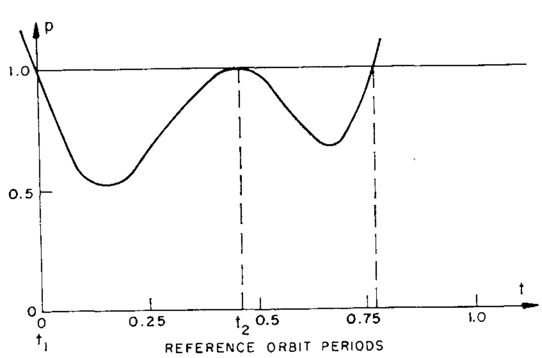

Fig. 6 Primer magnitude for optimal two-impulse with final coast.

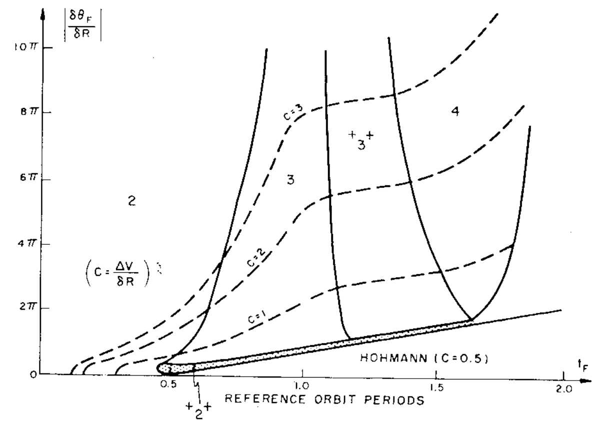

Fig. 7 Reachable final state variations. Optimal multiple impulse solutions.

VI. Summary of Multiple-Impulse Results

Figure 7 displays the essential results for optimal fixed-time multiple-impulse rendezvous between close coplanar circular orbits. An example application of this Figure is given on page 825 of Ref. 2. These linearized results, compare favorably with the Earth-Mars solutions presented in Ref. 12. These results can also be interpreted in terms of the information which would naturally be available at $t = 0$, namely the difference in terminal orbit radii, δR, and the initial target phase angle, β, (see Fig. 12). Using Eq. (54) one can show that[10]

$$\beta = \delta\theta_F + \tfrac{3}{4}t_F\delta R \tag{52}$$

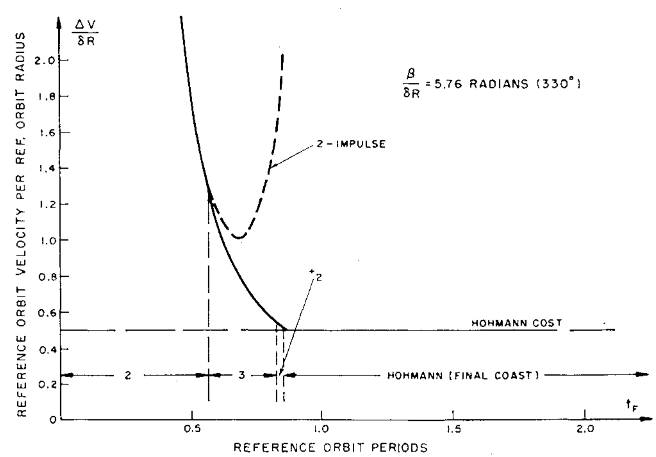

Fig. 8 Optimal coast as a function of transfer time for given initial conditions.

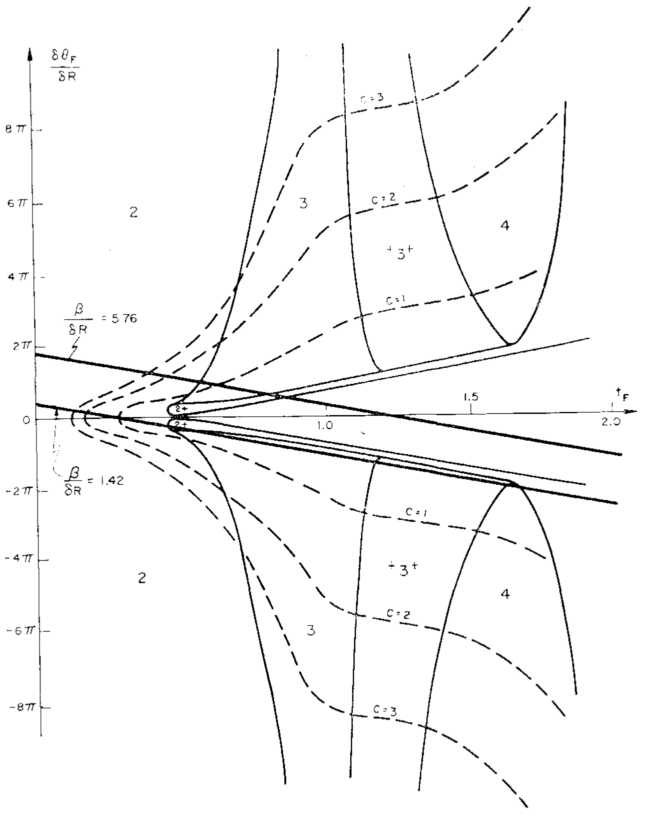

Fig. 9 Reachable state variations for given value of $\beta/\delta R$ (see Figs. 8 and 10).

For an example $\beta/\delta R$, Fig. 8 displays the cost of the rendezvous as a function of the transfer time. The solid curve represents the cost of the transfer; the numbers beneath the curve denote the number of impulses required. The curve shown in Fig. 8 corresponds to a slice through the plane of reachable final state variations as shown in Fig. 9.

For the $\beta/\delta R$ shown in Fig. 8, the two-impulse transfer is optimal for short transfer times. For t_F larger than 0.55 reference orbit periods, three-impulses are optimal. The dashed curve shows the two-impulse cost, which reaches a minimum and then rapidly increases due to the singularity at $t_F = 1$. The savings in fuel of the optimal transfer over the two-impulse transfer is approximately 20% at this minimum and increases with increasing transfer time as shown. For a small range of transfer times slightly less than 0.86, two impulses with initial coast are optimal. At $t_F = 0.86$ the absolute minimum Hohmann transfer is available. Since the time in-flight for this transfer is 0.5, this requires an initial coast period of duration 0.36. This corresponds to the time necessary to allow the correct geometrical phasing of the target for a Hohmann transfer. For $t_F > 0.86$, the optimal transfer is the Hohmann transfer with final coast of duration ($t_F - 0.86$) reference periods. Note that the fuel cost of the optimal transfer monotonically decreases with increasing transfer time.

Figure 10 shows an analogous curve for a different value of $\beta/\delta R$; its representation in Fig. 9 is also shown. In this case at $t = 0$ the opportunity for a Hohmann transfer is past and will not recur for a synodic period of the terminal orbits. However, the optimal cost curve approaches the Hohmann cost as the transfer time increases (Fig. 10). As in Fig. 8 two impulses are optimal for short transfer times. Between the regions of two- and three-impulse optimal transfers is a region of two-impulse with final coast. In this region it is best to make the transfer corresponding to the minimum of the two-impulse curve and employ a coast in the final orbit, during which the cost does not change. In a similar manner,

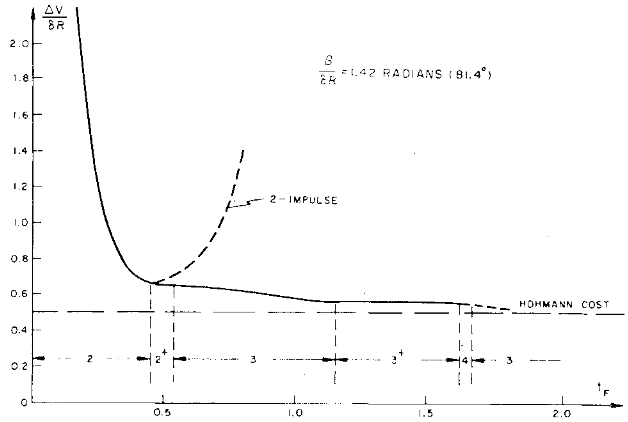

Fig. 10 Optimal coast as a function of transfer time for given initial conditions.

three-impulse with final coast solutions lie between the optimal three- and four-impulse solutions.

An Example Three-Impulse Rendezvous

To gain physical insight into multiple-impulse transfers Fig. 11 compares an optimal three-impulse rendezvous with the nonoptimal two-impulse rendezvous for the same boundary conditions and transfer time. As shown, the first impulse for both transfers is essentially tangential to the initial orbit. The two-impulse transfer intercepts the target with an appreciable radial velocity component; fuel must be expended to cancel this component to effect rendezvous. In contrast, the optimal three-impulse transfer arrives tangential to the final orbit. The mid-course velocity change, $\mathbf{\Delta V}_2$, causes the vehicle to intercept the target tangentially rather than fall inward toward the initial orbit, as shown by the dashed orbit continuation. For this specific example the nondimensional fuel cost for the optimal transfer is 0.63 compared with 1.06 for the two-impulse transfer. This com-

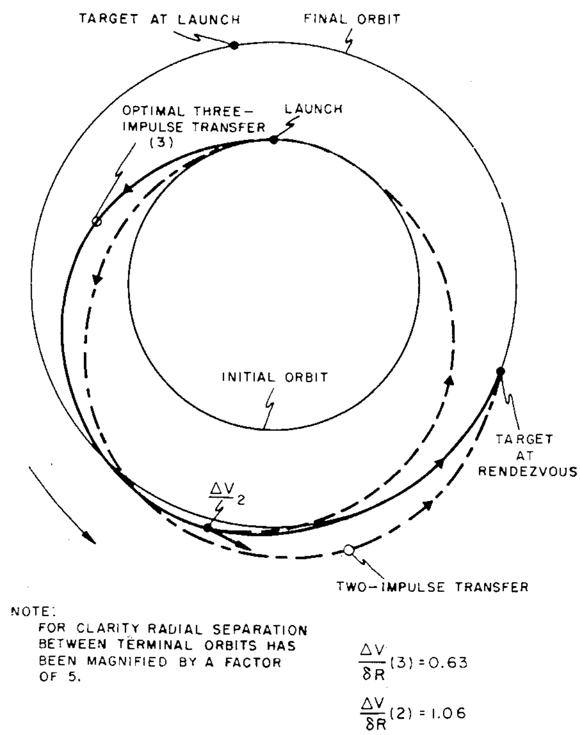

Fig. 11 Optimal three-impulse transfer.

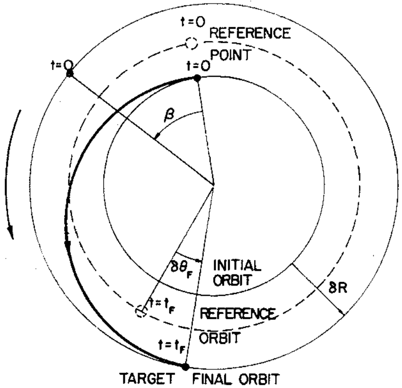

Fig. 12 Definition of state variables.

parison corresponds to Fig. 10 for a transfer time of 0.75, a time for which the two-impulse transfer is fairly uneconomical.

Appendix

The planar state variation in the vicinity of a circular reference orbit of radius a is conveniently coordinatized in terms of the variation in radial position, δr, and the variation in circumferential position, $a\delta\theta$, where $\delta\theta$ is the variation n central angle. The resulting state variation is (written in transposed form as a row vector),

$$\delta \mathbf{x}^T = [\delta \mathbf{r}^T \delta \mathbf{v}^T] = [\delta r \; a\delta\theta \; \delta\dot{r} \; a\delta\dot{\theta}] \qquad \text{(A1)}$$

In units such that a is unity and the reference orbital period is 2π, this state variation can be treated as dimensionless, with position variations being measured in fractions of reference orbital radius and velocity variations in fractions of reference orbital velocity.

As shown in Ref. 7 the state variation of a body in circular orbit of radius $1 + \delta a$ coplanar with the unit radius circular reference orbit is

$$\delta \mathbf{x}^T(t) = [\delta a \; \delta\theta(t) \; 0 \; -\tfrac{3}{4}\delta a] \qquad \text{(A2)}$$

For circle-to-circle coplanar transfer, it is convenient to choose the reference orbit to lie half-way between the terminal orbits. Let δR denote the difference between final and initial terminal orbit radii and choose the initial variation in central angle, $\delta\theta_0$, to be zero (Fig. 12). In this figure $t = 0$ corresponds to $\tau = \tau_0$ and $t = t_F$ corresponds to $\tau = \tau_F$ (i.e., $\tau - \tau_0 = \omega t$, where $\omega = 1$ in the chosen units).

The initial and final nondimensional state variations are then given by

$$\delta \mathbf{x}_0^T = [-\delta R/2 \quad 0 \quad 0 \quad \tfrac{3}{4}\delta R] \qquad \text{(A3)}$$

$$\delta \mathbf{x}_F^T = [\delta R/2 \,\vdots\, \delta\theta_F \quad 0 \quad -\tfrac{3}{4}\delta R] \qquad \text{(A4)}$$

To express $\widetilde{\delta \mathbf{x}}_H$ of Sec. IV in these coordinates, one notes that $\tau_F - \tau_H = \tau_H - \tau_0 (t_F = 2t_H)$, by definition. Using Eqs. (55) and (56) $\delta \mathbf{x}_F$ of Eq. (1) is given by

$$\widetilde{\delta \mathbf{x}}_F^T = [\delta R \,\vdots\, \delta\theta_F - \tfrac{3}{4}\delta R t_F \,\vdots\, 0 \,\vdots\, -\tfrac{3}{2}\delta R] \qquad \text{(A5)}$$

Since $\widetilde{\delta \mathbf{x}}_H = \Phi_{HF}\widetilde{\delta \mathbf{x}}_F$ and $t_H = t_F/2$

$$\widetilde{\delta \mathbf{x}}_H^T = [\delta R \quad \delta\theta_F \quad 0 \quad -\tfrac{3}{2}\delta R] \qquad \text{(A6)}$$

References

[1] Edelbaum, T. N., "How Many Impulses?" *Astronautics and Aeronautics*, Vol. 5, No. 11, Nov. 1967, pp. 64–69.

[2] Gobetz, F. W. and Doll, J. R., "A Survey of Impulsive Trajectories," *AIAA Journal*, Vol. 7, No. 5, May 1969, pp. 801–834.

[3] Robinson, A. C., "A Survey of Methods and Results in the Determination of Fuel-Optimal Space Maneuvers," Paper 68-091, Sept. 1968, American Astronautical Society.

[4] Lion, P. M. and Handelsman, M., "Primer Vector on Fixed-Time Impulsive Trajectories," *AIAA Journal*, Vol. 6, No. 1, Jan. 1968, pp. 127–132.

[5] Marec, J., "Rendez-vous Impulsionnels Optimaux de Longue Durée entre Orbites Quasi-Circulaires Proches, Coplanaires ou non," presented at the Colloquium on Advanced Problems and Methods for Space Flight Optimization, Liege, 1967.

[6] Marec, J., "Multi-impulse Optimal Rendez-Vous of Medium Duration, between Near-circular, Coplanar, Close Orbits," Paper AD47, Oct. 1968, International Astronautical Federation.

[7] Prussing, J. E., "Optimal Four-Impulse Fixed-Time Rendezvous in the Vicinity of a Circular Orbit," *AIAA Journal*, Vol. 7, No. 5, May 1969, pp. 928–935.

[8] Lawden, D. F., *Optimal Trajectories for Space Navigation*, Butterworths, London, 1963.

[9] Gantmacher, F. R., *The Theory of Matrices*, Vol. I, Chelsea, New York, 1960, p. 66.

[10] Prussing, J. E., "Optimal Multiple-Impulse Orbital Rendezvous," Sc. D. thesis, Sept. 1967, Dept. of Aeronautics and Astronautics, Massachusetts Institute of Technology.

[11] Steffan, K. F., "Orbital Guidance," *Guidance and Control of Aerospace Vehicles*, edited by C. T. Leondes, McGraw-Hill, New York, 1963, p. 462 ff.

[12] Minkoff, M. and Lion, P. M., "Optimal Multi-Impulse Rendezvous Trajectories," Paper AD-143, Oct. 1968, International Astronautical Federation.